B. Sc. part-II (Sub) Diff. Calculus 10-07-2020

Q.1. State and prove the Euler's Theorem on partial differentiation of homogeneous function of two independent variables.

Sol. Statement:→ Let $f(x,y)$ be a homogeneous function of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

Proof: Let $f(x,y) = Ax^{\alpha_1}y^{\beta_1} + Bx^{\alpha_2}y^{\beta_2} + Cx^{\alpha_3}y^{\beta_3} + \cdots$ —(1)

where, $\alpha_1+\beta_1 = \alpha_2+\beta_2 = \alpha_3+\beta_3 = \cdots = n$ —(2)

and A, B, C ... are constants i.e. independent of x and y.

Diff. (1) partially with respect to x, we get

$$\frac{\partial f}{\partial x} = A\alpha_1 x^{\alpha_1-1}y^{\beta_1} + B\alpha_2 x^{\alpha_2-1}y^{\beta_2} + C\alpha_3 x^{\alpha_3-1}y^{\beta_3} + \cdots$$

$$\therefore x\frac{\partial f}{\partial x} = A\alpha_1 x^{\alpha_1}y^{\beta_1} + B\alpha_2 x^{\alpha_2}y^{\beta_2} + C\alpha_3 x^{\alpha_3}y^{\beta_3} + \cdots \quad —(3)$$

Again, diff (1) partially with respect to y, we get

$$\frac{\partial f}{\partial y} = A\beta_1 x^{\alpha_1}y^{\beta_1-1} + B\beta_2 x^{\alpha_2}y^{\beta_2-1} + C\beta_3 x^{\alpha_3}y^{\beta_3-1} + \cdots$$

$$\therefore y\frac{\partial f}{\partial y} = A\beta_1 x^{\alpha_1}y^{\beta_1} + B\beta_2 x^{\alpha_2}y^{\beta_2} + C\beta_3 x^{\alpha_3}y^{\beta_3} + \cdots \quad —(4)$$

Adding (3) and (4), we get

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = Ax^{\alpha_1}y^{\beta_1}(\alpha_1+\beta_1) + Bx^{\alpha_2}y^{\beta_2}(\alpha_2+\beta_2) + Cx^{\alpha_3}y^{\beta_3}(\alpha_3+\beta_3) + \cdots$$

$$= Ax^{\alpha_1}y^{\beta_1}n + Bx^{\alpha_2}y^{\beta_2}n + Cx^{\alpha_3}y^{\beta_3}.n + \cdots$$ by virtue of (2)

$$= n(Ax^{\alpha_1}y^{\beta_1} + Bx^{\alpha_2}y^{\beta_2} + Cx^{\alpha_3}y^{\beta_3} + \cdots)$$

$= nf(x,y)$, by virtue of (1)

Proved.

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Problems based on partial derivatives

Q.2. If $u = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$, prove that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$.

Soln: We have, $u = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$

$$\Rightarrow u = x^2(y-z)+y^2(z-x)+z^2(x-y) \quad \text{--- (1)}$$

Differentiating partially with respect to x, keeping y and z constants, we get

$$\frac{\partial u}{\partial x} = 2x(y-z)+y^2(-1)+z^2(1)$$
$$= 2xy-2zx-y^2+z^2$$

Similarly, $\frac{\partial u}{\partial y} = x^2.1+2y(z-x)+z^2(-1)$
$$= x^2+2yz-2xy-z^2$$

and $\frac{\partial u}{\partial z} = x^2(-1)+y^2.1+2z(x-y)$
$$= -x^2+y^2+2zx-2yz$$

$$\therefore \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$$

Q.3. Show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$ when $u=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)$

Soln: We have, $\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\frac{1}{y}+\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)$

$$= \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\cdot\frac{-x}{y^2}+\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x}$$

$$= \frac{-x}{y\sqrt{y^2-x^2}}+\frac{x}{x^2+y^2}$$

$$\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{x}{\sqrt{y^2-x^2}}-\frac{xy}{x^2+y^2}-\frac{x}{\sqrt{y^2-x^2}}+\frac{xy}{x^2+y^2}=0$$